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# Time-dependent releases of solute in parallel flows

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A superposition of $\delta$-function inputs is used to investigate the approach to the steady state, oscillatory discharges, and a Gaussian pulse for high-Péclet-number parallel shear flows. Illustrative results are given for open-channel flow with a logarithmic velocity profile.

## 1. Introduction

Barton (1983) pointed out that there are two natural ways in which dispersion from a time-dependent input can be investigated: by a Fourier transform in time, or by a superposition of $\delta$-function inputs. He explored the viability of the first alternative, following upon the work of Brinkman (1950), Carrier (1956), Philips (1963*a*, *b*) and Chatwin (1973). The complexity of the mathematical structure revealed by Barton (1983) led him to conclude 'that it may not be desirable or optimal to solve the time-dependent injection problem by Fourier decomposition in time'.

Here we explore the viability of the second alternative, i.e. a superposition of $\delta$-function inputs. The approach to a steady state has been analysed in this way by Gill & Sankarasubramanian (1972). However, the complicated temporal structure of their $\delta$-function solution makes other problems inaccessible. Guided by the work of Tsai & Holley (1978) and of Chatwin (1980), Smith (1985) has given an alternative representation for the $\delta$-function solution, with a comparatively simple temporal structure. This alternative representation renders the superposition integrals analytically tractable, and here we are able to investigate oscillatory discharges and a Gaussian pulse as well as the approach to the steady state.

## 2. $\delta$-function solution

For a sudden release at $t = 0$, Smith (1985) posed the representation

$$c = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma}\exp\left(-\tfrac{1}{2}\xi^2\right)\left\{a^{(0)} + \sum_{m=3}^{\infty}\frac{a^{(m)}}{\sigma^m}\,\mathrm{He}_m(\xi)\right\}, \tag{2.1a}$$

with

$$\xi = \{t - T\}/\sigma. \tag{2.1b}$$

Here $a^{(0)}(x,y,z)$ is the integral with respect to time of the concentration (the total exposure), $T(x,y,z)$ is the temporal centroid, and $\sigma^2(x,y,z)$ temporal variance as observed at the location $(x,y,z)$. The Hermite polynomials $\mathrm{He}_m$ are defined recursively:

$$\mathrm{He}_0 = 1,\quad \mathrm{He}_1 = \xi,\quad \mathrm{He}_{m+2} = \xi\mathrm{He}_{m+1} - (m+1)\,\mathrm{He}_m. \tag{2.2}$$

In a Gaussian approximation we retain just the leading term.

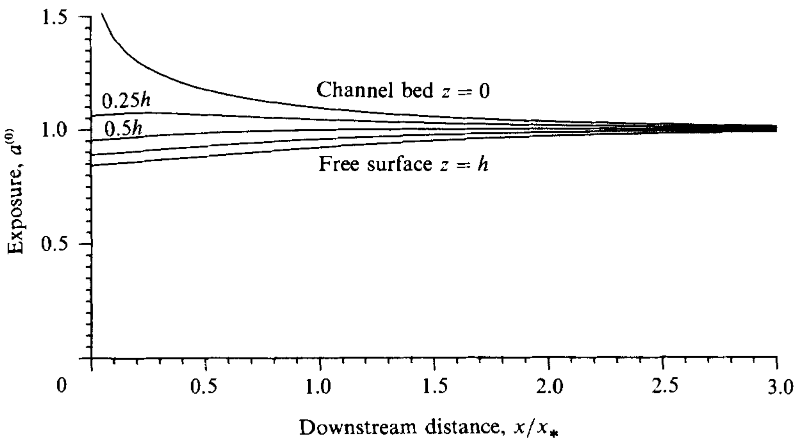


FIGURE 1. The exposure $a^{(0)}$ experienced at heights 0, $0.25h$, $0.5h$, $0.75h$, and $h$ above the bed for a uniform unit discharge in logarithmic open-channel flow. For a uniform unit surge in discharge rate, $a^{(0)}$ is the eventual steady-state concentration. The mixing length $x_*$ corresponds to about 20 water depths downstream.

For a plane parallel, high-Péclet-number flow with longitudinal velocity $u(y,z)$ and transverse diffusivity $\kappa$ the coefficients $a^{(0)}$, $T$, $\sigma^2$ satisfy the equations

$$u\,\partial_x a^{(0)} - \boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla} a^{(0)}) = q\delta(x), \tag{2.3a}$$

$$u\,\partial_x (a^{(0)}T) - \boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}(a^{(0)}T)) = a^{(0)}, \tag{2.3b}$$

$$u\,\partial_x (a^{(0)}\sigma^2) - \boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}(a^{(0)}\sigma^2)) = 2a^{(0)}\kappa(\boldsymbol{\nabla}T)^2. \tag{2.3c}$$

$\boldsymbol{\nabla}$ denotes the transverse gradient operator $(0, \partial_y, \partial_z)$ and $q(y,z)$ is the cross-stream profile of the source discharge situated at $x = 0$. For impermeable boundaries we have

$$0 = \kappa\boldsymbol{n}\cdot\boldsymbol{\nabla} a^{(0)} = \kappa\boldsymbol{n}\cdot\boldsymbol{\nabla}(a^{(0)}T) = \kappa\boldsymbol{n}\cdot\boldsymbol{\nabla}(a^{(0)}\sigma^2) \quad \text{on} \quad \partial A. \tag{2.3d}$$

If the diffusivity is not the same for $y$ and $z$, then the scalar $\kappa$ needs to be replaced by a tensor $\kappa_{ij}$. For moderate Péclet numbers it would be necessary to include the effects of longitudinal diffusion. This could be achieved by reinterpreting $\boldsymbol{\nabla}$ as being the three-dimensional gradient operator $(\partial_x, \partial_y, \partial_z)$, but at a considerable loss of mathematical simplicity (i.e. (2.3$a$, $b$, $c$) would become elliptic instead of being parabolic).

Smith (1985) gives asymptotic solutions at small distances:

$$a^{(0)} = \frac{q}{u} + \ldots, \quad T = \frac{x}{u} + \ldots, \quad \sigma^2 = \tfrac{2}{3}x^3\kappa\frac{(\boldsymbol{\nabla}u)^2}{u^5} + \ldots, \tag{2.4}$$

and at large distances downstream (far enough that there is thorough mixing across the flow):

$$a^{(0)} = \frac{\bar{q}}{\bar{u}} + \ldots, \quad T = \frac{x}{\bar{u}} + \ldots, \quad \sigma^2 = \frac{2xD}{\bar{u}^3} + \ldots. \tag{2.5}$$

Overbars indicate cross-sectional average values, and $D$ is the shear dispersion coefficient. These large-distance asymptotes, with the absence of $y, z$-dependence, correspond to the classical Gaussian distribution (Taylor 1953).

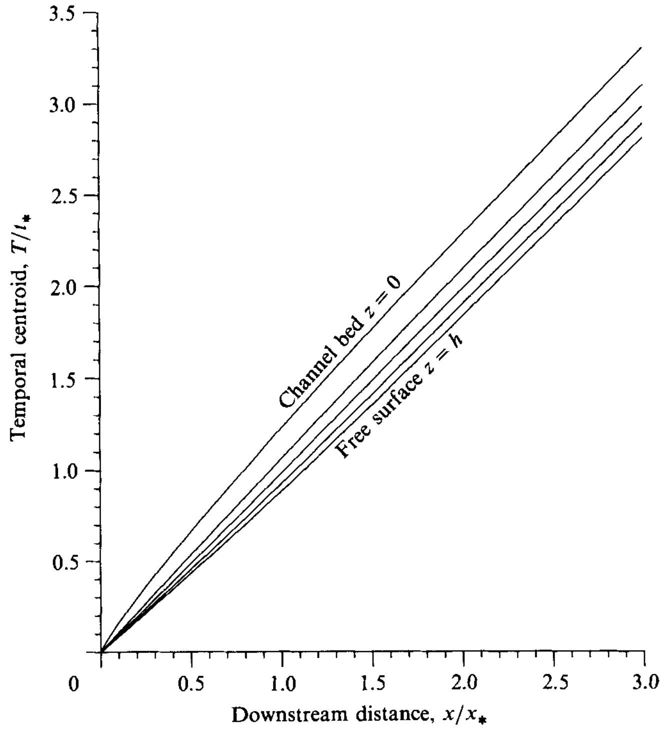


FIGURE 2. The centroid time of arrival $T$ as a function of downstream distances at heights 0, $0.25h$, $0.5h$, $0.75h$, and $h$ above the bed for a uniform discharge in logarithmic open-channel flow. For an oscillatory discharge with angular frequency $\omega$, the phase lag at different heights above the bed is given by $\omega T$. So there are marked phase differences for $\omega$ greater than $5/t_*$.

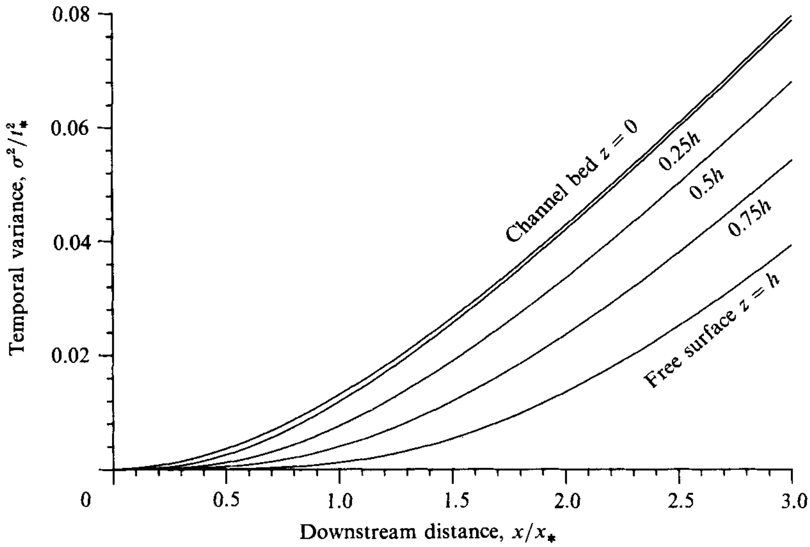


FIGURE 3. The temporal variance $\sigma^2$ as a function of distance downstream of a uniform discharge in logarithmic open-channel flow. For an oscillatory discharge with angular frequency $\omega$, the decay exponent at different heights above the bed is $-\frac{1}{2}\omega^2\sigma^2$. So, there is strong depth dependence for $\omega$ greater than $5/t_*$.

For intermediate distances, Smith (1985) gives series representations for $a^{(0)}$, $a^{(0)}T$ and $a^{(0)}\sigma^2$ in terms of advection–diffusion eigenmodes. The particular case of a uniform discharge in logarithmic open-channel flow is detailed in §7 of the present paper, and is used in all the illustrative examples. The mixing length $x_*$ is defined to be the distance in which vertical concentration differences (with no $x$-dependence) decay by a factor of e = 2.718. The mixing time $t_*$ is the corresponding time

$$t_* = \bar{u}x_*. \tag{2.6}$$

Figures 1, 2, 3 show that for several mixing lengths downstream of the discharge, there is marked depth-dependence of the exposure, centroid time, and temporal variance.

## 3. Convolution integral

For a continuous discharge with strength $f(t)$, but with fixed cross-stream profile $q(y, z)$, the concentration can be written as a superposition of $\delta$-function solutions (Gill & Sankarasubramanian 1972, equation 4; Barton 1983, equation 8.2). The resulting extension of (2.1) can be written

$$c = \int_{-\infty}^{\infty} f(t-T-\sigma\xi)\,\frac{\exp(-\frac{1}{2}\xi^2)}{(2\pi)^{\frac{1}{2}}}\left\{a^{(0)} + \sum_{m=3}^{\infty}\frac{a^{(m)}}{\sigma^m}\,\mathrm{He}_m(\xi)\right\}\mathrm{d}\xi. \tag{3.1}$$

If, instead of the discharge rate, it was the initial concentration that was specified, then a different superposition integral would be required.

By causality the full series (2.1) must be identically zero for $t < 0$, i.e. for $\xi < -T/\sigma$. So, in principle, the lower limit in the convolution integral (3.1) could be set as $-T/\sigma$. However, any truncation of the series (2.1) does not preserve the causality property. It is to allow for this (spurious) future influence that the lower limit of the integral (3.1) is extended to $-\infty$.

The rapid decay with $\xi$ enables us to estimate that if we make a Gaussian approximation and retain just the $a^{(0)}$ term, then the relative error as regards causality is of order

$$\int_{-\infty}^{-T/\sigma}\frac{\exp(-\frac{1}{2}\xi^2)}{(2\pi)^{\frac{1}{2}}}\mathrm{d}\xi \approx \frac{1}{(2\pi)^{\frac{1}{2}}}\frac{\sigma}{T}\exp\left(-\frac{T^2}{2\sigma^2}\right) \quad \text{for} \quad \sigma \ll T. \tag{3.2}$$

The quotient $T/\sigma$ tends to infinity in the limits of small and of large $x$. Also, from the numerical results for $T$ and for $\sigma^2$ shown in figures 3(*a*, *b*), 7 and 4(*a*, *b*), 8 of Smith (1985) we can infer that $T/\sigma$ has a minimum of about 4. Thus, in making a Gaussian approximation the relative error as regards causality would be of order $3\times 10^{-5}$ or less.

The figures presented in this paper concern logarithmic open-channel flow. The disparity between the bulk velocity $\bar{u}$ and the turbulent friction velocity $u_*$ leads to large values of $T/\sigma$ (in excess of 10 for all $x$). So, any errors as regards causality are negligible.

## 4. Approach to the steady state

As a first illustrative example, we follow Gill & Sankarasubramanian (1972) and consider a discharge at $x = 0$, which is switched on at time $t = 0$ with unit strength

$$f(t) = 1 \quad \text{for} \quad t > 0. \tag{4.1}$$

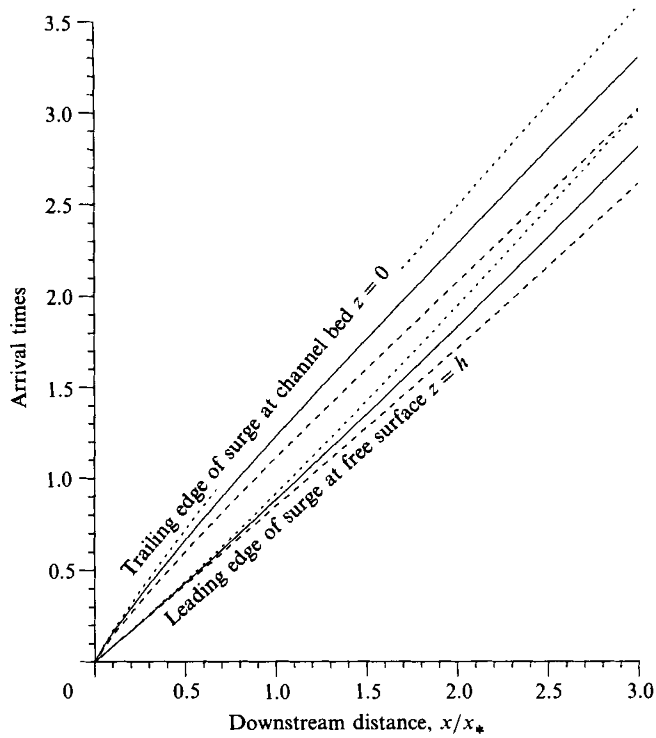


FIGURE 4. The times of arrival at the free surface and at the bed of the leading edge $(T-\sigma)/t_*$ (----), the midpoint $T/t_*$ (——), and the trailing edge $(T+\sigma)/t_*$ (......), for a uniform surge in discharge strength in logarithmic open-channel flow.

The approach to the steady state is given by

$$c = a^{(0)}\left\{1-\tfrac{1}{2}\operatorname{erfc}\left(\frac{(t-T)}{\sigma\sqrt{2}}\right)\right\}-\frac{\exp}{(2\pi)^{\frac{1}{2}}}\left(-\frac{(t-T)^2}{2\sigma^2}\right)\sum_{m=3}^{\infty}\frac{a^{(m)}}{\sigma^m}\,\mathrm{He}_{m-1}\left(\frac{t-T}{\sigma}\right). \quad (4.2)$$

The error-function structure agrees with the small- and large-$x$ asymptotes derived by Gill & Sankarasubramanian (1972, equations 43, 46). The fact that $a^{(0)}(x,y,z)$ is the steady-state solution can be deduced directly from the field equation (2.3$a$). Thus, in the present context, figure 1 gives the steady-state solution.

At times $\sigma$ before and after the centroid time $T$, the concentration has reached 0.16 and 0.84 of the ultimate steady-state values. Figure 4 shows this spread of times for the concentrations observed at the free surface and at the bed in logarithmic open-channel flow. Close to the discharge the surge of concentration arrives earlier and is much more abrupt at the free surface than at the bed. However, at larger distances downstream there has been more mixing across the flow, so there begins to be some overlap between the surges as observed at the free surface and at the bed. When the concentration is *not* independent of $x$, the marked velocity difference between the free surface and the bed considerably delays the approach to uniform concentrations across the flow. Thus, the Taylor (1953) limit (2.5) only applies after ten or more mixing lengths downstream of the discharge.

## 5. Oscillatory discharges

As a second example, we follow Barton (1983) and Chatwin (1973) and consider a harmonic discharge strength

$$f(t) = \cos \omega t. \tag{5.1}$$

This can be regarded as being one harmonic in a Fourier decomposition of a periodic discharge strength. The necessary integrals involving $\mathrm{He}_m(\xi)$ are tabulated by Erdelyi *et al.* (1954, p. 289, equation 6), and lead to the neat result

$$c = \exp\left(-\tfrac{1}{2}(\omega\sigma)^2\right)\left\{a^{(0)} + \sum_{m=2}^{\infty} (-1)^m a^{(2m)} \omega^{2m}\right\} \cos(\omega(t-T)). \tag{5.2}$$

At different depths the values of $\sigma^2$ shown in figure 3 differ by as much as $t_*^2/25$. So, the value of $\omega = 5/t_*$ demarcates whether the exponential decay is similar or varies markedly across the flow.

For high frequencies ($\omega$ greater than $5/t_*$) the downstream penetration of concentration fluctuations is extremely sensitive to $\sigma^2$. From the small-$x$ asymptote (2.4) we infer (in agreement with Chatwin 1973) that the distribution of concentration becomes exponentially small except in a region centred on the fastest part of the flow (i.e. where $\sigma^2$ has its minimum).

The phase lines follow the temporal centroid $T(x, y, z)$. So, from the small-$x$ asymptote (2.4), we recover another of Chatwin's (1973) results, that for high frequencies the transport velocity will be near the maximum fluid velocity. Also, the phase difference across the flow can be well in excess of $\pi$.

For low frequencies ($\omega$ less than $5/t_*$) the concentration fluctuations penetrate to large values of $\sigma$. Thus, we can use the large-$x$ asymptotes for the phase and decay. As deduced by Chatwin (1973), there is weaker variation of concentration across the flow (amplitude or phase) and the effective transport velocity is $\bar{u}$.

## 6. Gaussian pulse

The table of Hermite polynomial integrals given by Erdelyi *et al.* (1954, §16.5) gives us a copious supply of tractable examples. In particular, their equation (17) permits us to deal with a Gaussian pulse

$$f(t) = \frac{\exp}{\rho(2\pi)^{\frac{1}{2}}}\left(-\frac{t^2}{2\rho^2}\right), \tag{6.1}$$

$$c = \frac{1}{(2\pi(\sigma^2+\rho^2))^{\frac{1}{2}}} \exp\left(-\frac{(t-T)^2}{2(\sigma^2+\rho^2)}\right)\left\{a^{(0)} + \sum_{m=3}^{\infty} \frac{a^{(m)}}{(\sigma^2+\rho^2)^{m/2}} \mathrm{He}_m\left(\frac{t-T}{(\sigma^2+\rho^2)^{\frac{1}{2}}}\right)\right\}. \tag{6.2}$$

This is the same as adding together the $\delta$-function variance $\sigma^2$ and the source variance $\rho^2$ in the basic solution (2.1*a*), or displacing the $x$-axis downwards in figure 3.

At small distances downstream the effect is to remove the regions of high concentration, particularly near the velocity maximum where $\sigma^2$ is anomalously small (see figure 3). Far downstream, where $T$ and $\sigma^2$ grow linearly with distance, it is as if there were a virtual $\delta$-function source at

$$x = -\frac{\bar{u}^3\rho^2}{2D}, \quad t = -\frac{\bar{u}^2\rho^2}{2D}. \tag{6.3}$$

## 7. Uniform discharge in logarithmic open-channel flow

The figures which have been used to illustrate the structure of the solutions all concern the special case

$$\left.\begin{aligned} q = 1, \quad u = \bar{u} + \frac{u_*}{k}\left(1 + \ln\left(\frac{z}{h}\right)\right), \\ \kappa = k u_* h\left(1 - \frac{z}{h}\right)\left(\frac{z}{h}\right), \quad 0 < z < h. \end{aligned}\right\} \tag{7.1}$$

Here $\bar{u}$ is the bulk velocity, $u_*$ the friction velocity, $k$ is von Kármán's constant (in this paper we use the value 0.4), $h$ the water depth, and $z$ the vertical coordinate. For simplicity, there is no $y$-diffusion and the problem is only two-dimensional. We shall make considerable use of the fact that $u_*$ is typically much smaller then $\bar{u}$.

If we are to use the series solutions for $a^{(0)}$, $a^{(0)}T$ and $a^{(0)}\sigma^2$ derived by Smith (1985), then our first task is to determine the advection–diffusion eigenmodes:

$$\nabla \cdot (\kappa \nabla \phi_n) + \mu_n u \phi_n = 0, \tag{7.2a}$$

with

$$\kappa \boldsymbol{n} \cdot \nabla \phi_n = 0 \quad \text{on} \quad \partial A, \tag{7.2b}$$

$$\overline{u\phi_n^2} = \bar{u}, \quad \overline{u\phi_n \phi_m} = 0 \quad \text{for} \quad m \neq n. \tag{7.2c, d}$$

Two-term approximations are given by

$$\mu_n = n(n+1)\frac{k u_*}{h\bar{u}}\left[1 + \frac{u_*}{k\bar{u}}\left\{-\frac{2n}{2n+1} + \frac{1}{2}\sum_{j=1}^{n}\frac{1}{(j-\frac{1}{2})j}\right\} + \ldots\right], \tag{7.3a}$$

$$\begin{aligned} \phi_n = (2n+1)^{\frac{1}{2}}\Bigg[&P_n\left(2\frac{z}{h} - 1\right) + \frac{u_*}{k\bar{u}}\frac{(-1)^n}{n(n+1)} \\ &+ \frac{u_*}{k\bar{u}} n(n+1) \sum_{m \neq n} \frac{\operatorname{sgn}(m-n)(-1)^{m+n+1}(2m+1)^{\frac{1}{2}}}{(m-n)^2(m+n+1)} P_m\left(2\frac{z}{h} - 1\right) \\ &+ \frac{u_*}{2k\bar{u}} P_n\left(2\frac{z}{h} - 1\right)\left\{-\frac{2n}{2n+1} + \frac{1}{2}\sum_{j=1}^{n}\frac{1}{(j-\frac{1}{2})j}\right\} + \ldots\Bigg]. \end{aligned} \tag{7.3b}$$

Here $P_m$ denotes the Legendre polynomial of degree $m$. The need to go beyond a first approximation in the small parameter $u_*/\bar{u}$ stems from the occurrence of the coefficients

$$\overline{\phi_n} = \frac{(2n+1)^{\frac{1}{2}}(-1)^n}{n(n+1)}\frac{u_*}{k\bar{u}} + \ldots, \tag{7.4a}$$

$$\overline{\phi_n^2} - 1 = \frac{u_*}{k\bar{u}}\left\{\frac{-2n}{2n+1} + \frac{1}{2}\sum_{j=1}^{n}\frac{1}{(j-\frac{1}{2})j}\right\} + \ldots, \tag{7.4b}$$

$$\overline{\phi_n \phi_m} = \frac{u_*}{k\bar{u}}\frac{(2n+1)^{\frac{1}{2}}(2m+1)^{\frac{1}{2}}(-1)^{n+m}}{|m-n|(m+n+1)} + \ldots \quad \text{for} \quad m \neq n. \tag{7.4c}$$

The dots indicate neglected terms of order $(u_*/\bar{u})^2$.

Keeping the leading $z$-dependent terms, we can simplify the general results given by Smith (1985) to

$$\bar{u}a^{(0)} = 1 + \frac{u_*}{k\bar{u}}\sum_{n=1}^{\infty}\frac{(2n+1)(-1)^n}{n(n+1)}P_n\left(2\frac{z}{h} - 1\right)\exp(-\mu_n x), \tag{7.5a}$$

$$\bar{u}T = x + \frac{h}{k^2} \sum_{n=1}^{\infty} \frac{(2n+1)(-1)^n}{n^2(n+1)^2} P_n\left(2\frac{z}{h}-1\right)[1-\exp(-\mu_n x)], \tag{7.5b}$$

$$\begin{aligned}\bar{u}^2\sigma^2 = {} & 2\frac{x}{\bar{u}}\frac{hu_*}{k^3} \sum_{n=1}^{\infty} \frac{2n+1}{n^3(n+1)^3} - 2\frac{h^2}{k^4} \sum_{n=1}^{\infty} \frac{2n+1}{n^4(n+1)^4}[1-\exp(-\mu_n x)] \\ & + 2\frac{h^2}{k^4} \sum_{n=1}^{\infty} f_n(x) P_n\left(2\frac{z}{h}-1\right) \\ & - \frac{h^2}{k^4}\left\{\sum_{n=1}^{\infty} \frac{(2n+1)(-1)^n}{n^2(n+1)^2} P_n\left(2\frac{z}{h}-1\right)[1-\exp(-\mu_n x]\right\}^2,\end{aligned} \tag{7.5c}$$

where the coefficients $f_n(x)$ are given by the summations

$$\begin{aligned}f_n = {} & \frac{(2n+1)(-1)^n}{n^3(n+1)^3}[1-(\mu_n x+1)\exp(-\mu_n x)]\left\{-\frac{2n}{2n+1}+\frac{1}{2}\sum_{j=1}^{n}\frac{1}{(j-\frac{1}{2})j}\right\} \\ & + \frac{(2n+1)(-1)^n}{n(n+1)} \sum_{m \neq n} \frac{2m+1}{m^2(m+1)^2|m-n|(m+n+1)} \\ & \times\left\{1-\frac{\mu_m \exp(-\mu_n x)-\mu_n \exp(-\mu_m x)}{\mu_m-\mu_n}\right\}.\end{aligned} \tag{7.5d}$$

For the $n = 1$ mode we can define an e-folding distance $x_*$, and a corresponding mixing time $t_*$:

$$x_* = \frac{h\bar{u}}{2ku_*}, \quad t_* = \frac{x_*}{\bar{u}} = \frac{h}{2ku_*}. \tag{7.6a, b}$$

In particular, if we specify

$$\bar{u} = 16u_*, \quad k = 0.4, \tag{7.6c, d}$$

then the mixing length $x_*$ is twenty times the water depth. The results shown in figures 1–4 are made non-dimensional with respect to these natural scales $x_*, t_*$. For other flows the results can be expected to be qualitatively similar.

Some authors use

$$t_c = \frac{h^2}{\bar{\kappa}} = \frac{6h}{ku_*} \tag{7.7}$$

as the characteristic mixing time. This is twelve times larger than $t_*$. The advantage of the present definition is the transferability of the results to other flows.

## 8. Concluding remarks

The neatness of the results (4.2), (5.2), (6.2) for the approach to the steady state, oscillatory discharges, and for a Gaussian pulse are strong evidence for the efficacy of the superposition of $\delta$-function inputs. In the one-term (Gaussian) approximation the temporal concentration distribution has the same functional form at all positions. The three key ingredients $a^{(0)}(x,y,z)$, $T(x,y,z)$ and $\sigma^2(x,y,z)$ can be interpreted as being measures of the amplitude of the concentration, the time lag from the discharge to the observation position, and the extent of the temporal averaging. For high-Péclet-number flows (i.e. with longitudinal diffusion dominated by shear dispersion) the equations for $a^{(0)}$, $T$ and $\sigma^2$ are amenable to exact and asymptotic solution.

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